Name of College - S.S. College J. Bad.
Deptt - Mathematics
Topic → Hyperbolic Functions
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Class - B.Sc I (Hons)

Study Material

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1. Definitions of Hyperbolic Functions :→

Notation - Hyperbolic sin of z → sinh z
Hyperbolic cos of z - cosh z
Hyperbolic tan of z → tanh z etc.

Here z is a real or complex

sinh z is defined as $\dfrac{e^{z}-e^{-z}}{2}$

$\cosh z = \dfrac{e^{z}+e^{-z}}{2}$

$\tanh z = \dfrac{e^{z}-e^{-z}}{e^{z}+e^{z}}$

$\operatorname{cosech} z = \dfrac{1}{\sinh z}$

$\operatorname{sech} z = \dfrac{1}{\cosh z}$

$\coth z = \dfrac{1}{\tanh z}$.

Relation between circular functions and hyperbolic functions

Since we know $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$.

$\Rightarrow \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Replace θ by ix and using $i^2=-1$ $\quad -i^2 = 1$

$\cos ix = \dfrac{e^{iix}+e^{-iix}}{2}$

$= \dfrac{e^{-x}+e^{x}}{2}$

$= \dfrac{e^{x}+e^{-x}}{2} = \cosh x$

$\Rightarrow \sin ix = i\sinh x$

Also $\sin ix = \dfrac{e^{iix}-e^{-iix}}{2i}$

$= \dfrac{e^{-x}-e^{+x}}{2i}$

$\Rightarrow i\sin ix = \dfrac{e^{-x}-e^{+x}}{2} = \sinh x$

$\cancel{\Rightarrow \sin ix = \dfrac{1}{i}\sinh x = -\dfrac{e^{x}-e^{-x}}{2}\cdot i^2 = -i\sinh x}$

$\Rightarrow i\sin ix = -\dfrac{(e^{x}-e^{-x})}{2}$ ...

Since $\sin ix = i\sinh x$

$\cos ix = \cosh x$

$\Rightarrow \tan ix = i\tanh x$

Conversion of circular functions into Hyperbolic Functions

Thus $\sin ix = i\sinh x$

$\cos ix = \cosh x$

$\tan ix = i\tanh x$

Conversion of hyperbolic Functions in to circular Functions.

$\sinh x = \frac{1}{i}\sin ix$

$= -i\sin ix$

$\cosh x = \cos ix$

$\tanh x = -i\tan ix$

Importantant Formulae for hyperbolic Functions:—

1. $\cosh^2 x - \sinh^2 x = 1$

$1+\sinh^2 x = \cosh^2 x$

$\cosh^2 x - 1 = \sinh^2 x$

for $\cosh^2 x - \sinh^2 x$

$$= \left[\frac{e^x + e^{-x}}{2}\right]^2 - \left[\frac{e^x - e^{-x}}{2}\right]^2$$

$$= \frac{1}{4}\left[(e^x + e^{-x})^2 - (e^x - e^{-x})^2\right]$$

using $(a+b)^2 - (a-b)^2 = 4ab.$

$$= \frac{1}{4}4.e^x.e^{-x}$$

$$= 1$$

Now $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow 1 - \tanh^2 x = \operatorname{sech}^2 x \Rightarrow \operatorname{sech}^2 x + \tanh^2 x = 1$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\operatorname{sech}^2 x = 1 - \tanh^2 x$

Again Since $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \coth^2 x - 1 = \operatorname{cosech}^2 x$

$\Rightarrow \coth^2 x ~~+~~ \operatorname{cosech}^2 x = 1$

$\coth^2 x - 1 = \operatorname{cosech}^2 x$

$1 + \operatorname{cosech}^2 x = \coth^2 x$

Expansion of $\sinh x$ and $\cosh x$

Since we have $\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \cdots$

$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \cdots$

Now $\sinh x = -i \sin ix$

$$= -i\left[ix - \frac{i^3x^3}{\lfloor 3} + \frac{i^5x^5}{\lfloor 5} - \cdots\right]$$

$$= -i^2x + \frac{i^4x^3}{\lfloor 3} - \frac{i^6x^5}{\lfloor 5} - \cdots$$

using $i^2 = -1$ $\quad i^4 = 1$ $\quad i^{4n+1} = i$ $\quad i^{4n+2} = -1$ $\quad i^{4n+3} = -i$

Where $n = 1, 2, 3 \cdots$

$$= x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots$$

$$\cosh x = \cos ix = 1 - \frac{i^2x^2}{\lfloor 2} + \frac{i^4x^4}{\lfloor 4} - \frac{i^6x^6}{\lfloor 6} + \cdots$$

$$= 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^6}{\lfloor 6} + \cdots$$

Periods of hyperbolic Functions $\longrightarrow$

$\sinh x$ and $\cosh x$ are periodic functions whose periods are imaginary and equal to $2n\pi i$

but $\tanh x$ is also periodic function whose periods are imaginary and Equal to $in\pi$

Thus

Hyperbolic Function	Period.	Nature of Period.
$f(x) = \sinh x$	$2n\pi i$	Imaginary
$= \cosh x$	$2n\pi i$	Imaginary
$= \tanh x$	$n\pi i$	Imaginary